

Symmetry Arguments and the Role They Play in Using Gauss' Law

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Symmetry plays a very important role in science in general, and physics in particular. Arguments based on symmetry can often simplify calculations by allowing us to take shortcuts (e.g., not computing some component of the electric field because we can prove using symmetry that it must be zero). Symmetry also plays a vital role in much of the current theoretical physics research. Aside from a few general principles (and perhaps a few parameters which need to be fit to experimental data), the only assumptions that are made in these modern “Quantum Field Theories” or “String Theories” are the symmetries that the universe is assumed to obey.

It is important to understand symmetry arguments — not only how they work, but also *why* they work (i.e., why such arguments can be considered to be rigorously valid, and not “cheating”). In this short note, we discuss symmetry arguments used in calculating the electric field due to various charge distributions, and in particular, the role played by symmetry in the use of Gauss' Law to calculate the electric field due to highly symmetric charge distributions.

Underlying principles

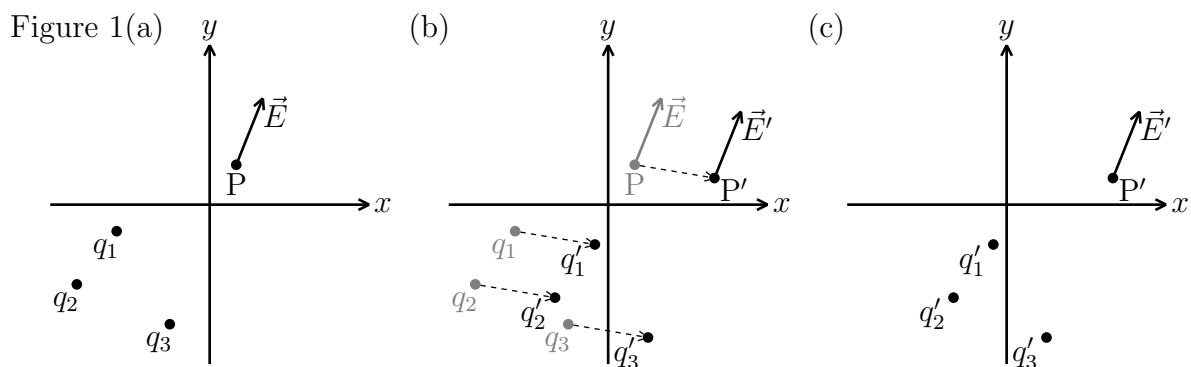
The validity of symmetry arguments involved in the calculation of the electric field due to some charge distribution is based on two principles:

1. For any given charge distribution and given point P, there can only be one possible value for the electric field at P due to the given charge distribution.
2. If the charge distribution is moved — either by translation, rotation around some fixed axis, mirror reflection across some plane, or any combination of these operations¹ — then the electric field must “move with the charge distribution”.

The first principle should be self-evident. We can calculate the electric field at P using Coulomb's Law and superposition once we are given the charge distribution and the point P where we are evaluating the electric field.

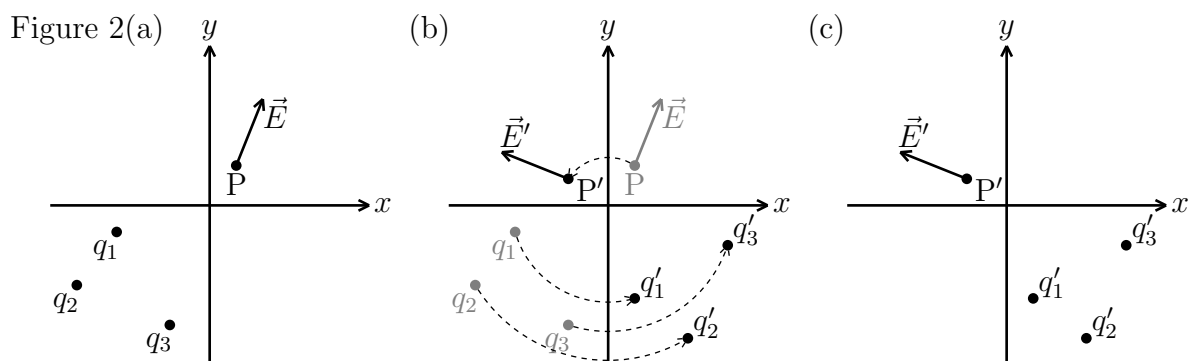
To help illustrate the second principle, consider the following charge distribution and point P illustrated in Figure 1(a) below.

¹The common property shared by all three of these operations (translation, rotation, and mirror reflection) is that they are all *isometric* — they preserve all distances and angles among the various charges.



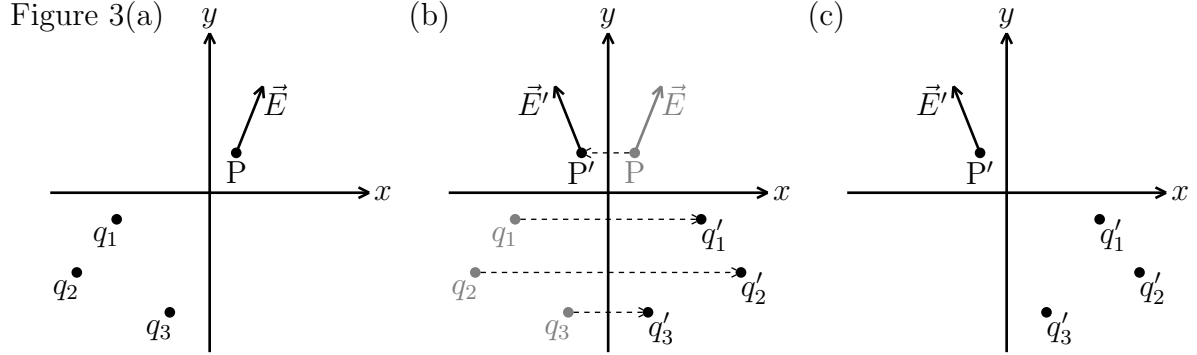
Suppose we calculate the electric field at P due to this charge distribution (presumably using Coulomb's Law and superposition), and the result is as indicated in Figure 1(a). Now consider what happens when we move all three charges by the same displacement vector (Fig. 1(b)). If we also move the point P by the same displacement, it seems reasonable that the electric field \vec{E}' at the translated point P' due to the translated charge distribution should be equal to the original electric field \vec{E} at P due to the original charge distribution. That is precisely what happens: $\vec{E}' = \vec{E}$.²

The same idea can be applied to rotation and mirror reflection. In Figure 2 below, we apply a 90° counterclockwise rotation around the z-axis (perpendicular to the plane of the page) to our original charge distribution. After rotating the point P as well, we conclude that \vec{E}' , the electric field at the rotated point P' due to the rotated charge distribution, should be related to the electric field \vec{E} at P due to the original charge distribution. In this case, \vec{E}' is not *equal* to \vec{E} , but is the result of rotating \vec{E} by the same 90° angle (in particular, the *magnitudes* of \vec{E} and \vec{E}' are equal).



Likewise, in Figure 3 below, we mirror-reflect the charge distribution and the point P through the y-z plane (changing the sign of all x coordinates but not changing any y or z coordinates). The resulting electric field \vec{E}' should be the mirror reflection of the original electric field \vec{E} (once again, the magnitudes will be equal).

²One way to see that this is reasonable is to ask what would happen if, instead of moving the charges and the point P, you moved the coordinate system in the opposite direction.

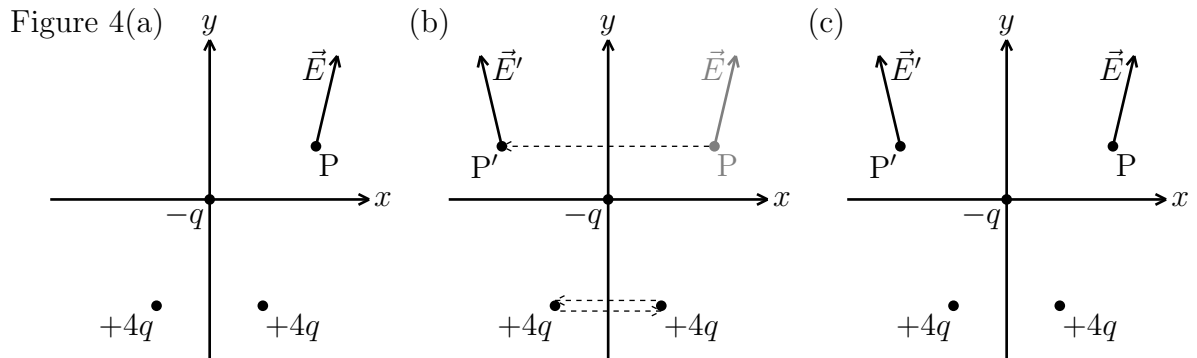


In all three cases, the fact that the electric fields before and after the transformation are related as described above can be proved rigorously by Coulomb's Law and superposition, with the help of some linear algebra.

Symmetric charge distributions

In each case above, we were able to relate the electric field at some point P due to some charge distribution to the electric field at some *other* point P' due to a *different* charge distribution. However, we were unable to learn anything about the electric field at P due to the original charge distribution, nor were we able to compare the electric fields at two different points due to the *same* charge distribution. This is because the charge distribution considered in the previous section was not symmetric under translation, rotation, or mirror reflection. It turns out that when we consider *symmetric* charge distributions, we will be able to gain more useful information by moving the charges around.

Consider the symmetric charge distribution shown in Figure 4 below.

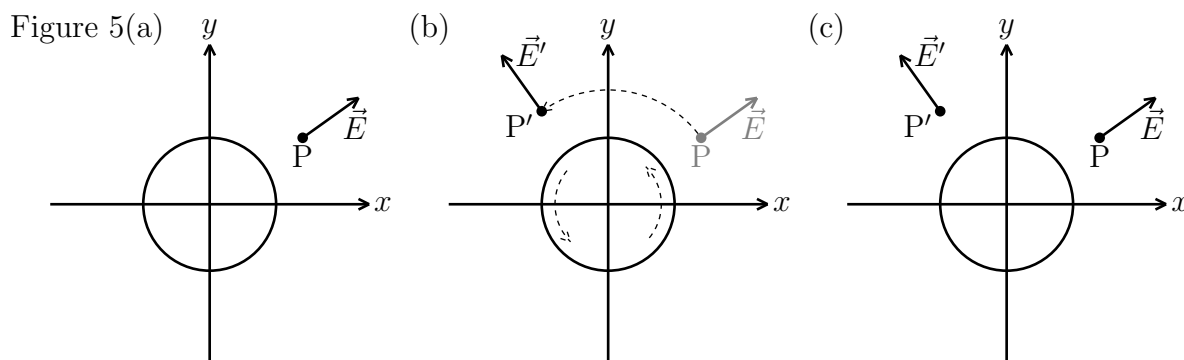


If we mirror-reflect the charge distribution and the point P across the y - z plane, as we did in Figure 3 in the previous section, we will find that the electric field \vec{E}' at the mirror-reflected point P' due to the mirror-reflected charge distribution is, itself, a mirror reflection of the electric field \vec{E} at the original point P due to the original charge distribution.

However, in this case, the original charge distribution is symmetric under the mirror reflection we just performed. How does this help us? When we mirror-reflect the original symmetric charge distribution, *the result is exactly the same charge distribution again* (see Fig. 4(b)). Therefore, the electric fields \vec{E} and \vec{E}' at the points P and P', respectively, are due to the *same charge distribution*.

Evidently, when the charge distribution is symmetric, we can use a symmetry argument, such as the one given above, to compare the electric field at two distinct (but symmetrically related) points due to that same charge distribution. If we were to apply the same argument to a number of points in the vicinity of this charge distribution, we would find the entire electric field pattern to be symmetric. If we were to draw an electric field line diagram for these charges, that diagram would have to be symmetric as well. *The electric field must obey the same symmetries that are obeyed by the charge distribution*. This is not the result of a hand-waving argument — *this principle is based on rigorous mathematical reasoning*.

For another example, this one involving rotation, consider the circular (not spherical) charge distribution shown below in Figure 5 (the circle and the point P both lie in the x - y plane).



If we rotate the charge distribution counterclockwise by 90° , we conclude that the electric field \vec{E}' at P' due to the rotated charge distribution must be the result of a 90° counterclockwise rotation of the electric \vec{E} at P due to the original charge distribution. Now — this is the key — *if the circular charge distribution is uniform, or otherwise symmetric under 90° rotation*, then the rotated charge distribution will coincide with the original charge distribution, and the electric fields \vec{E} and \vec{E}' will both be due to the same (original) charge distribution. In fact, an arbitrary-angle rotation symmetry (which is obeyed by the circular rod if its linear charge density is uniform) allows us to compare the electric field at P to the electric field at all points in the x - y plane which are the same distance from the center of the circle as the point P itself. *However, if the linear charge density of the circle is not uniform and is not otherwise 90° -rotation symmetric, then the comparison of electric fields at the points P and P' will not work, as those fields would be due to different charge distributions*.

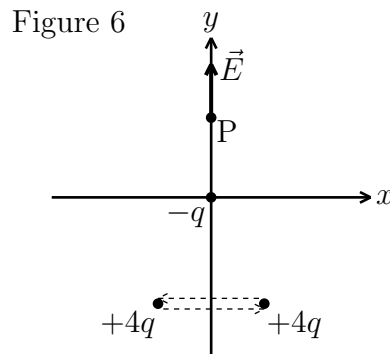
For an example involving translational symmetry, consider a uniformly charged, straight, infinite rod. The translational symmetry of the rod will allow a comparison of the electric

field at points P and P' which are related by a displacement pointing in the same direction as the rod. The details are left to the reader (you will see this argument presented in class when Gauss' Law is used to calculate the electric field due to an infinite rod or cylinder).

Symmetrically located points

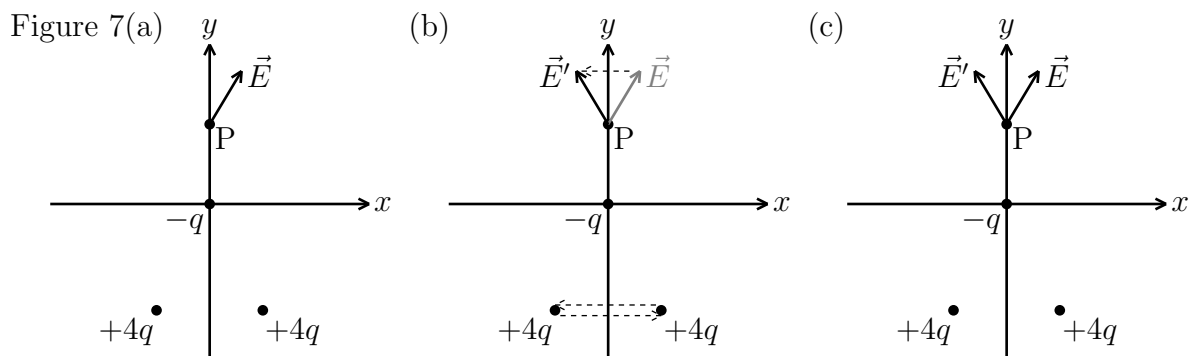
As we have seen from last section, a symmetry argument can be used to compare the electric field at two distinct points due to a symmetric charge distribution. The same type of argument can also be used to gain information about the electric field at a single point, provided that the point is symmetrically located.

For our first example, consider the three-charge symmetrical charge distribution. Suppose we wish to evaluate the electric at a point P located on the y -axis. Our gut instinct tells us that the electric field must point in a symmetrical direction (either the $+y$ or $-y$ direction), and must therefore resemble Figure 6, shown below.



However, gut instincts can sometimes be wrong, so can we actually *prove* that the electric field must point in one of those two directions? It turns out that we can, using the same type of symmetry argument that we used in the last section.

Suppose for a moment that the electric field is not symmetrical, in that it has a non-zero x -component, as shown in Figure 7(a) below.



If we mirror-reflect the charge distribution and the point P across the y - z plane, as we have already done several times, we find that the electric field \vec{E}' at the reflected point P'

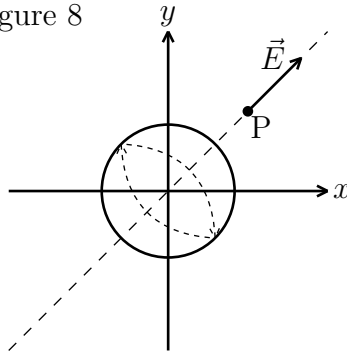
due to the reflected charge distribution must be the result of mirror-reflecting the original electric field \vec{E} at the original point P due to the original charge distribution. However, both the charge distribution *and the point P where we are evaluating the field* are symmetrical, and therefore P' coincides with P and the reflected charge distribution coincides with the original charge distribution. It follows that the two electric fields, \vec{E} and \vec{E}' , must both be the electric field at the *same* point due to the *same* charge distribution. According to the first principle, there can only be one possible value for the electric field at this point, so \vec{E} and \vec{E}' must be equal. However, this is not the case if \vec{E} is not symmetrical (see Fig. 7(c)). It follows that the electric field *cannot* be asymmetrical, as shown in Fig. 7(a), and must therefore be symmetrical, as shown in Fig. 6.

There is another argument that the electric field at P must be symmetrical, with which the reader may be more familiar. The electric field due to each individual charge is calculated using Coulomb's Law and then added together according to the Superposition Principle to get the total electric field. For every charge on the left side of the symmetry plane, there is a corresponding charge on the right side (e.g., the two $+4q$ charges in the present case). Because of the symmetric placement of those charges, and of the point P, the x components of the electric field contributions from the two charges must cancel. When all such electric field contributions are added together, *all* of the electric field x components cancel out, and so E_x must be zero.

There is nothing wrong with that argument. However, it is quite different from the symmetry argument we presented using Figure 7. In that figure, the vector \vec{E} is intended to represent the *total* electric field, presumably having been calculated by summing up contributions from *all* of the charges. We hypothesize that, despite the symmetry of the charge distribution and the point P, this asymmetric vector \vec{E} indeed represents the total electric field at P. We then use mirror reflection to conclude that the mirror-reflected vector \vec{E}' must *also* represent the total electric field at this same point due to the same charge distribution. We are in no way asserting that \vec{E} and \vec{E}' are to be combined in any way (e.g., by adding them) in order to produce the “actual” electric field at P. Instead, we reason that since \vec{E} is not equal to \vec{E}' , our original assumption that \vec{E} is the (total) electric field at P must have been false. *The actual electric field at P must coincide with its own mirror reflection*, an example of which is the electric field shown in Figure 6.

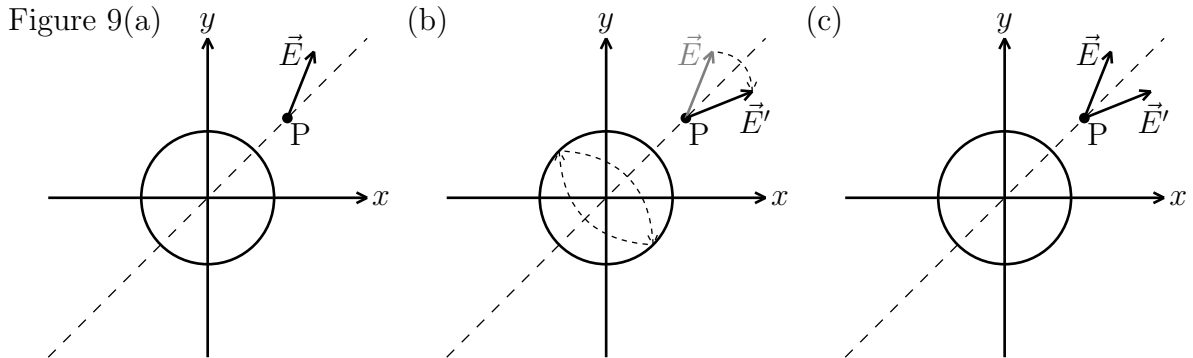
Another example of this same type of symmetry argument is the argument that the electric field due to a uniform circle of charge lying in the x - y plane must be radial at all points in the x - y plane, as shown in Figure 8 below.

Figure 8



To make the case, we take advantage of the 180° “flip” rotation symmetry of the circular charge around the dashed-line axis passing through the point P (the dashed line in the above figure lies in the x - y plane — it is not a perspective drawing of the z axis). The 180° rotation essentially “flips” the circle over like a pancake, moving the upper-left portion of the circle to the lower-right, and vice versa. The point P lies on the axis of rotation (the axis was chosen so that this was true), and thus is not affected by the rotation.

Suppose that the electric field were not radial, but instead were directed as shown in Figure 9, below.



Applying the flip rotation to the circular charge distribution and the point P leaves both unchanged, but causes the electric field \vec{E} to flip over to \vec{E}' . Since there can only be one value of the electric field at P due to the uniform circular charge distribution, \vec{E} would have to equal \vec{E}' . However, that cannot happen if \vec{E} is not radial, as shown in Figure 9(c). Therefore, the total electric field must in fact be radial, as shown in Figure 8 (note that the electric field in Figure 8 would be unaffected by the flip rotation).

Note that if the charge distribution did not obey this flip rotation symmetry (e.g., because the linear charge density were not uniform), the symmetry argument would fall apart — the electric fields \vec{E} and \vec{E}' , both at the point P, would be due to different charge distributions, and no contradiction results even if \vec{E} were not radial.

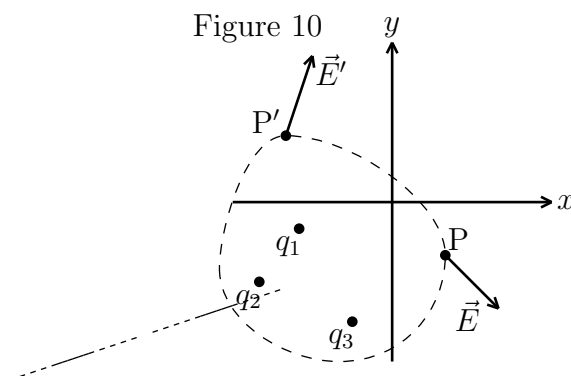
There is no translation-symmetric example appropriate for this section, since any translation will move a point P to a different point P'.

Gauss' Law applications

Gauss' Law asserts that the net electric flux (in the outward direction) through any closed surface must be proportional to the charge enclosed by that surface:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} .$$

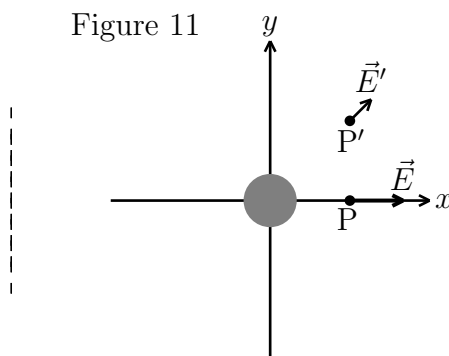
For example, in Figure 10 below, the electric flux through the indicated surface must be equal to $(q_1 + q_2 + q_3)/\epsilon_0$.



It should be pointed out that the *validity* of Gauss' Law has nothing to do with symmetry. Gauss' Law is still true, even when the charge distribution and the closed surface are not symmetric in any way. Where symmetry arguments come into play is when we attempt to use Gauss' Law to evaluate the electric field *at an individual point*.

Suppose we wanted to evaluate the electric field at P in Figure 10 using Gauss' Law. The problem is that the electric flux not only depends on the electric field at P, it also depends on the electric field at P', and at every other point on the Gaussian surface. Because of the lack of symmetry, there is no obvious relationship between the electric field at P and at P', and so no useful information about the electric field at individual points can be gained by using Gauss' Law to calculate $\oint \vec{E} \cdot d\vec{A}$.

Now consider a spherically symmetric charge distribution, illustrated in Figure 11 below.



Can Gauss' Law be used to calculate the electric field at P due to this charge distribution? The answer turns out to be yes, provided that we choose our Gaussian surface wisely. Is the

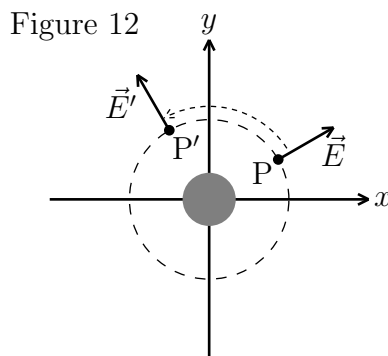
box depicted in Figure 11 a good choice? No it isn't. Once again, the electric flux through the box depends on the electric field at P, at P', and at every other point on the surface of the box. In order to have any chance of being able to evaluate the electric field at P from the integral $\oint \vec{E} \cdot d\vec{A}$, we need to be able to relate the electric fields at P and P', presumably by taking advantage of the rotation symmetry of the spherically symmetric charge distribution. However, the points P and P' are at different radii (i.e., different distances from the center of the charge distribution), and so it is impossible to rotate P into P'. There is no obvious relationship between the electric fields at P and P' (assuming we haven't already solved the problem by some other means), and so once again, we conclude that no useful information about the electric field at individual points can be gained by calculating $\oint \vec{E} \cdot d\vec{A}$ for the box.

Again, the *validity* of Gauss' Law is not being questioned — we may still conclude

$$\oint_{\text{box}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0} .$$

Knowing this, however, does not help us calculate the electric field at individual points, such as P. (By the way, can a symmetry argument be used to evaluate $\int \vec{E} \cdot d\vec{A}$ over just a single face of the box? Think about it.)

To calculate the electric field at P due to the spherically symmetric charge distribution, we need to be able to take advantage of the rotation symmetry of the charge distribution. *In general, this means that the Gaussian surface must have the same type of symmetry as the charge distribution.* The box proved to be a poor choice for the Gaussian surface because it had the wrong symmetry. Let's now consider the spherical Gaussian surface depicted in Figure 12 below.



Can we calculate the electric field at individual points (such as P) by applying Gauss' Law to the sphere? Yes, this will work. The reason it works is because the charge distribution *and the Gaussian surface* are both highly symmetrical — they both possess full (arbitrary-angle) rotation symmetry about any axis passing through the center of the charge distribution.

We may first apply the 180° “flip” rotation symmetry to conclude that the electric field at all points must be radially directed. The argument is exactly the same as for the uniform circle (see Figs. 8 and 9). The reason why this first step is necessary is that the electric flux only depends on one component of the electric field at any given point — namely the

component perpendicular to the Gaussian surface (the radial component, in the case where the Gaussian surface is a sphere). We need to find some other means to evaluate the electric field components which are parallel to the surface. In the present case, those non-radial components must be zero, by symmetry.

At this point, we can simplify the electric flux integral as follows:

$$\text{Flux} = \oint \vec{E} \cdot d\vec{A} = \oint E_r dA .$$

However, the flux integral still depends on the electric field not only at P, but also at P', and at all other points on the Gaussian sphere. How do we get past this problem?

We need to be able to relate the electric field at P and at P', and at all other points on the sphere. Since all points on the Gaussian sphere are at the same radius, we can rotate P to any one of the other points on the surface. For example, we can map P to P' using a 90° counterclockwise rotation around the z-axis. We may use exactly the same symmetry argument given in Figure 5 for the circle to conclude that \vec{E}' is the result of rotating \vec{E} through the same 90° angle that we used to rotate P into P'. In particular, \vec{E} and \vec{E}' must have the same radial component.

It follows that the radial component in the flux integral above is constant *and can be brought out of the integral*. The flux integral can be further simplified:

$$\text{Flux} = \oint E_r dA = E_r \oint dA = E_r(4\pi r^2) .$$

The value of E_r that we pulled out of the integral is the radial component of the electric field *at any individual point on the Gaussian surface, including the point P*. Setting the flux equal to $Q_{\text{encl}}/\epsilon_0$ (Gauss' Law) and solving for E_r yields

$$E_r = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{encl}}}{r^2} .$$

This formula can be applied to any spherically symmetric charge distribution.

It is important that you understand the argument above, and in particular, the large role that symmetry played in the calculation of the electric field. Gauss' Law itself was involved only briefly (towards the end when we set the flux equal to $Q_{\text{encl}}/\epsilon_0$). Most of the calculation was based on simplifying the electric flux integral with the help of symmetry arguments. In general, we require two symmetry arguments: (1) to determine the direction of the electric field (in order to establish that all but one component of the electric field is zero, and allow us to simplify the dot product $\vec{E} \cdot d\vec{A}$), and (2) to prove that the non-zero component of the electric field is constant over the surface, thereby allowing us to pull that component out of the flux integral. Only by pulling the electric field component out of the integral can we then solve for the electric field *at individual points* when we finally set the flux equal to $Q_{\text{encl}}/\epsilon_0$.

Other examples of Gauss' Law calculations will be presented in class.